

A note on three dimensional good sets

K. GOWRI NAVADA

Department of Mathematics, Periyar University, Salem - 636011, India

E-mail: gnavada@yahoo.com

2000 Mathematics Subject Classification: primary 60A05, 47A35, secondary 28D05, 37Axx

Abstract:. We show that as in the case of n - fold Cartesian product for $n \geq 4$, even in 3-fold Cartesian product, a related component need not be full component.

Key words. Good set; full set; full component; related component; geodesic; boundary of a good set.

Introduction and Preliminaries The purpose of this note is to answer two questions about good sets raised in [3] and [4] for the case $n = 3$.

Let X_1, X_2, \dots, X_n be nonempty sets and let $\Omega = X_1 \times X_2 \times \dots \times X_n$ be their Cartesian product. We will write \vec{x} to denote a point $(x_1, x_2, \dots, x_n) \in \Omega$.

For each $1 \leq i \leq n$, Π_i denotes the canonical projection of Ω onto X_i .

A subset $S \subset \Omega$ is said to be *good*, if every complex valued function f on S is of the form:

$$f(x_1, x_2, \dots, x_n) = u_1(x_1) + u_2(x_2) + \dots + u_n(x_n), \quad (x_1, x_2, \dots, x_n) \in S, \quad (1)$$

for suitable functions u_1, u_2, \dots, u_n on X_1, X_2, \dots, X_n respectively ([3], p. 181).

For a good set S , a subset $B \subset \bigcup_{i=1}^n \Pi_i S$ is said to be a *boundary set of S* , if for any complex valued function U on B and for any $f : S \rightarrow \mathbb{C}$ the equation (1) subject to

$$u_i|_{B \cap \Pi_i S} = U|_{B \cap \Pi_i S}, \quad 1 \leq i \leq n,$$

admits a unique solution. For a good set there always exists a boundary set ([3], p. 187).

A subset $S \subset \Omega$ is said to be *full*, if S is maximal good set in $\Pi_1 S \times \Pi_2 S \times \dots \times \Pi_n S$.

A set $S \subset \Omega$ is full if and only if it has a boundary consisting of $n - 1$ points ([3], Theorem 3, page 185).

If a set S is good, maximal full subsets of S form a partition of S . They are called *full components* of S ([3], p. 183).

Two points \vec{x}, \vec{y} in a good set S are said to be *related*, denoted by $\vec{x} R \vec{y}$, if there exists a finite subset of S which is full and contains both \vec{x} and \vec{y} . R is an equivalence relation, whose equivalence classes are called *related components* of S . The related components of S are full subsets of S (ref. [3]).

First we prove that when the dimension $n = 3$, a full component need not be a related component, by giving an example of a full set with infinitely many related components.

Consider a countable set T which consists of the following points:

$$\vec{a}_1 = (x_1, x_2, x_3)$$

$$\vec{a}_2 = (y_1, y_2, x_3)$$

$$\vec{a}_3 = (y_1, x_2, z_3)$$

$$\vec{a}_4 = (\alpha_1, \alpha_2, \alpha_3)$$

$$\begin{aligned}
\vec{a}_5 &= (\alpha_4, \alpha_5, \alpha_3) \\
\vec{a}_6 &= (\alpha_1, \alpha_5, z_3) \\
\vec{a}_7 &= (\alpha_4, \alpha_2, x_3) \\
\vec{a}_8 &= (x_1, y_2, \alpha_3) \\
\vec{a}_9 &= (\alpha_6, \alpha_7, \alpha_8) \\
\vec{a}_{10} &= (\alpha_9, \alpha_{10}, \alpha_8) \\
\vec{a}_{11} &= (\alpha_6, \alpha_{10}, \alpha_3) \\
\vec{a}_{12} &= (\alpha_9, \alpha_7, x_3) \\
\vec{a}_{13} &= (x_1, \alpha_2, \alpha_8) \\
&\dots \\
&\dots \\
\vec{a}_{5n-1} &= (\alpha_{5n-4}, \alpha_{5n-3}, \alpha_{5n-2}) \\
\vec{a}_{5n} &= (\alpha_{5n-1}, \alpha_{5n}, \alpha_{5n-2}) \\
\vec{a}_{5n+1} &= (\alpha_{5n-4}, \alpha_{5n}, \alpha_{5n-7}) \\
\vec{a}_{5n+2} &= (\alpha_{5n-1}, \alpha_{5n-3}, x_3) \\
\vec{a}_{5n+3} &= (x_1, \alpha_{5n-8}, \alpha_{5n-2}) \\
&\dots \\
&\dots
\end{aligned}$$

Call the first three points of T as D_0 and for $n \geq 1$, let D_n denote the first $3 + 5n$ points of S . Let $A_0 = D_0$ and for $n \geq 1$ let $A_n = D_n \setminus D_{n-1}$. Then it is easy to see that every D_n is good and has three point boundary. All the three points of the boundary of D_n cannot come from the coordinates of points in D_{n-1} : because, if all of them occur as coordinates in D_{n-1} , they form a boundary for D_{n-1} . Given any function f on D_n , there is a solution u_1, u_2, u_3 on D_{n-1} such that

$$f(w_1, w_2, w_3) = u_1(w_1) + u_2(w_2) + u_3(w_3), \quad (w_1, w_2, w_3) \in D_{n-1}.$$

But then $f(\vec{a}_{5n+3})$ fixes the value of $u_3(\alpha_{5n-2})$ by the following equation:

$$u_3(\alpha_{5n-2}) = f(\vec{a}_{5n+3}) - u_1(x_1) - u_2(\alpha_{5n-8})$$

When we substitute this value of $u_3(\alpha_{5n-2})$ in the remaining four points of A_n , we get a set of linearly dependent equations. This shows that the boundary of D_n contains at least one of the five coordinates, $\alpha_{5n-4}, \alpha_{5n-3}, \alpha_{5n-2}, \alpha_{5n-1}$ or α_{5n} , which are introduced in A_n . One can observe the following properties of the points in the set A_n : any k points of A_n has at least k coordinates introduced in A_n . (i.e, they do not occur as coordinates in D_{n-1}). If we take a singleton $\{\vec{a}_i\}$ in D_{n-1} , any set of k points of A_n has at least $(k+1)$ coordinates which do not occur as coordinates of \vec{a}_i .

T is good as every finite subset of T is good. It cannot have a boundary B with more than two points: If $|B| = 3$, we can choose a n sufficiently large such that all the three points of b occur as coordinates in D_{n-1} . Then B is a boundary of D_n which is not possible as observed above. If $|B| > 3$, we can choose n sufficiently large so that $k = |B \cap \cup_{i=1}^3 \Pi_i D_n| \geq 4$. Then these k points form a boundary of D_n which is again not possible. So the boundary of T consists of only two points which shows that T is full.

We prove that no finite subset A of T other than singleton is full: Set $|A \cap A_i| = k_i$ for $i \geq 0$. Let $i_1 < i_2 < \dots < i_l$ be such that $k_{i_j} \neq 0$ for $j = 1, 2, \dots, l$ and $k_i = 0$ for all other i . If $k_{i_1} > 1$, as no subset other than singleton of A_n is full, the set $A \cap A_{i_1}$ is not full. When we add the points of $A \cap A_{i_2}$ to $A \cap A_{i_1}$ (as we are adding k_{i_2} points) we will be adding at least k_{i_2} new coordinates. So the set $A \cap (A_{i_1} \cup A_{i_2})$ is not full. Similarly when we keep adding $A \cap A_{i_j}$ to the set $A \cap (\cup_{k < j} A_{i_k})$ the number of coordinates added is at least equal to the number of points added. So at each step $A \cap (\cup_{k \leq j} A_{i_k})$ is not full. In this way we get $A = A \cap (\cup_{k \leq l} A_{i_k})$ is also not full. If $k_{i_1} = 1$, in the first step when we add points of $A \cap A_{i_2}$ to the singleton set $A \cap A_{i_1}$ the new coordinates added is at least $k_{i_2} + 1$. So $A \cap (A_{i_1} \cup A_{i_2})$ is not full. In the remaining steps as we keep adding points from $A \cap A_{i_j}$, the number of coordinates added is at least equal to the number of points added. So in the end we get A is not full.

For any n , let $\vec{b}_n = (\alpha_{5n-1}, y_2, z_3)$ and consider the set $F_n = D_n \cup \vec{b}_n$. We show that the geodesic between the points \vec{a}_1 and \vec{a}_{5n+3} in F_n is the whole set F_n . To show that F_n is full, consider the matrix M_n whose rows correspond to the points $\vec{a}_2, \vec{a}_3, \dots, \vec{a}_{5n+3}, \vec{b}_n$ and columns correspond to the coordinates $y_1, y_2, z_3, \alpha_1, \alpha_2, \dots, \alpha_{5n}$. This is a $5n+3 \times 5n+3$ matrix:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 1 & 1 & 1 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & . & . & . & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & . & . & . & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

It has an inverse given by

[illegible]

This shows that F_n is full. To show that it is the geodesic between the points \vec{a}_1 and \vec{a}_{5n+3} in F_n , we show that any proper subset A of F_n containing these two points is not full. If possible suppose such a set A is full. Then A has to contain the point \vec{b}_n because no subset of D_n , other than singleton, is full.

Let $k = |F_n| - |A|$. As A is full there exists atleast k coordinates of points of F_n which donot occur as coordinates in the points of A . (Because otherwise adding these k points we get F_n and we will be adding less than k coordinates. If A is full then F_n cannot be good). Let S denote these k coordinates. The set S cannot contain $x_1, x_2, x_3, \alpha_{5n-1}, \alpha_{5n-2}, \alpha_{5n-8}, y_2$ and z_3 as these are used by the points of A . Among these k coordinates let k_i be the number which are introduced in $A_i, 0 \leq i \leq n$. For $i \geq 1$, we have $0 \leq k_i \leq 5$ and $k_0 = 0$ or 1 . If $0 < k_i < 5$ for some $i \geq 1$, (or if $k_0 = 1$ for $i = 0$) the k_i coordinates of S introduced in A_i are used in atleast $k_i + 1$ points of A_i . So if $k_0 = 1$ or $0 < k_i < 5$ for some $i \geq 1$, then more than k points of F_n cannot be in A which is a contradiction. In the case $k_0 = 0$ and $k_i = 0$ or $k_i = 5$ for $i \geq 1$, clearly there exists an $i \geq 1$ with $k_i = 5$. But in this case we have $k_{n-1} = k_n = 0$. If $k_i = 5$, then $A \cap A_i = \phi$ and if $k_i = 0$, then $A \cap A_i = A_i$. Let j be an index such that $k_j = 5$ and $k_{j+1} = 0$. Then $A \cap A_{j+1} = A_{j+1}$ which is a contradiction because A_{j+1} uses coordinates introduced in A_j which are not used by points of A . This shows A is not full.

It can be seen that the 5 rows of M_n^{-1} , from $(5m-1)th$ row to $(5m+3)rd$ row, have row sums bounded by $C_1 + C_2 \sum_{i=1}^m \frac{1}{2^i}$ for some constants C_1 and C_2 , independent of n . This shows that as in higher dimensions, in the three dimensional case also uniform boundedness of lengths of geodesics is not a necessary condition for boundedness of solutions of (1) for bounded function f .

Acknowledgement: I thank Prof. M G Nadkarni for suggesting the problem, fruitful discussions and encouragement.

References:

- [1] Cowsik R C, Kłopotowski A and Nadkarni M G, When is $f(x, y) = u(x) + v(y)$?, *Proc. Indian Acad. Sci. (Math. Sci.)* 109 (1999) 57-64.
- [2] Kłopotowski A, Nadkarni M G and Bhaskara Rao K P S, When is $f(x_1, x_2, \dots, x_n) = u_1(x_1) + u_2(x_2) + \dots + u_n(x_n)$?, *Proc. Indian Acad. Sci. (Math. Sci.)* 113 (2003) 77-86.
- [3] Kłopotowski A, Nadkarni M G and Bhaskara Rao K P S, Geometry of good sets in n -fold Cartesian products, *Proc. Indian Acad. Sci. (Math. Sci.)* 114 (2004) 181-197.
- [4] Nadkarni M G, Kolmogorov' s superposition theorem and sums of algebras, *The Journal of Analysis* vol. 12 (2004) 21-67.
- [5] Gowri Navada K, Some remarks on good sets, *Proc. Indian Acad. Sci. (Math. Sci.)* 114 (2004) 389-397.
- [6] Gowri Navada K, Some further remarks on good sets, to appear in *Proc. Indian Acad. Sci. (Math. Sci.)*.